

## The Bethe ansatz for $\text{AdS}_5 \times \text{S}^5$ bound states

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JHEP01(2009)005

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## The Bethe ansatz for $\text{AdS}_5 \times S^5$ bound states

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**ABSTRACT:** We reformulate the nested coordinate Bethe ansatz in terms of coproducts of Yangian symmetry generators. This allows us to derive the nested Bethe equations for arbitrary bound state string S-matrices. The bound state number dependence in the Bethe equations appears through the parameters  $x^\pm$  and the dressing phase only.

**KEYWORDS:** AdS-CFT Correspondence, Bethe Ansatz, Quantum Groups.

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## 1. Introduction and summary

The AdS/CFT correspondence [1] has been the subject of intensive research. This duality provides a powerful tool to study a plethora of interesting topics in theoretical physics. One of the best studied examples is  $\mathcal{N} = 4$  SYM gauge theory which is conjectured to be dual to superstring theory on  $\text{AdS}_5 \times \text{S}^5$ . However, even testing or proving the duality for these concrete models is a difficult problem.

A breakthrough in the understanding of this duality was the discovery of integrable structures. Integrability was found in  $\mathcal{N} = 4$  super Yang-Mills theory by the appearance of (integrable) spin chains describing the operator spectrum [2]. The classical string sigma model on  $\text{AdS}_5 \times \text{S}^5$  was also shown to be integrable [3]. If the full quantum theories also exhibit integrable structures, then that would constrain them considerably. For example, in scattering processes for integrable theories the set of particle momenta is conserved and scattering processes always factorize into a sequence of two-body interactions. This implies that in such theories the scattering information is encoded in the two-body S-matrix. Unfortunately, a full proof of integrability of both theories is currently still lacking, but nevertheless there is a lot of evidence hinting that integrability is a feature of the full quantum theories.

By assuming integrability one can make use of the S-matrix approach. This proved to be a powerful instrument to study the operator spectrum of  $\mathcal{N} = 4$  SYM. [2, 4–6]. This led to the conjecture of the “all-loop” Bethe equations describing the gauge theory asymptotic spectrum [7, 8]. For the  $\text{AdS}_5 \times \text{S}^5$  superstring, based on the knowledge of the classical

finite-gap solutions [9], a Bethe ansatz for the  $\mathfrak{su}(2)$  sector was proposed [10]. Finally, in both cases, exact two body S-matrices were found [8, 11]. These enabled the use of the Bethe ansatz [8, 12, 13], confirming the conjectured Bethe equations for physical states.

More precisely, the two body S-matrix (scattering fundamental multiplets) that appears in this approach, is almost completely fixed by symmetry. Both the asymptotic spectrum of  $\mathcal{N} = 4$  super Yang-Mills theory [5, 8] as well as the light-cone Hamiltonian [14–16] for the  $\text{AdS}_5 \times \text{S}^5$  superstring exhibit the same symmetry algebra; centrally extended  $\mathfrak{su}(2|2)$ . The requirement that the S-matrix is invariant under this algebra determines it uniquely up to an overall phase factor [8] and the choice of representation basis [11]. In a suitable local scattering basis, the S-matrix respects most properties of massive two-dimensional integrable field theory, like unitarity, crossing symmetry and the Yang-Baxter equation [11]. Furthermore, the (dressing) phase appeared to be a striking feature of the string S-matrix and it has been studied intensely, see e.g. [17–20]. By combining its expansion in terms of local conserved charges with crossing symmetry [21], one can find interesting solutions [17, 22], which incorporates string and gauge theory data.

Actually, apart from the multiplet of fundamental particles, the string sigma model also contains an infinite number of bound states [23]. These bound states fall into short (atypical) symmetric representations of the centrally extended  $\mathfrak{su}(2|2)$  algebra [24–26]. Of course, these states scatter via their own S-matrices. Recently a number of these bound states S-matrices have been found; they describe scattering processes involving fundamental multiplets and two-particle bound state multiplets [27] and the scattering of a fundamental multiplet with an arbitrary bound state multiplet [28]. However, invariance under centrally extended  $\mathfrak{su}(2|2)$  is not enough to fix all these S-matrices. One needs to impose the Yang-Baxter equation by hand in order to fix them up to a phase. The overall phase that still remains can be chosen to satisfy crossing symmetry [27].

Both the fundamental and the two particle bound state S-matrices were shown to exhibit a larger symmetry algebra<sup>1</sup> of Yangian type [29–32]. Moreover, as an alternative to the Yang-Baxter equation, the bound state S-matrices appear to be completely fixed (again up to an overall scale), by requiring invariance under Yangian symmetry [32]. This seems to indicate that the Yangian symmetry is restrictive enough to fix all bound state S-matrices and perhaps should be seen as the fundamental scattering symmetry.

The formulated asymptotic Bethe ansätze, following from the S-matrix approach, only describe the spectra in the infinite volume limit. However, for a full check of this particular case of the AdS/CFT correspondence, the complete spectra have to be computed and compared. Away from the asymptotic region, matters become more involved because wrapping interactions appear. One way to include these is Lüscher’s perturbative approach [33–36]. In Lüscher’s approach one deals with corrections coming from virtual particles that propagate around the compact direction. These virtual particles can be both fundamental particles and bound states. This method has proven quite successful since it recently allowed the computation of the full four-loop Konishi operator, including wrapping interactions [28, 37]. The result coincided with the gauge theory computation [38–40], providing an extremely non-trivial check of the correspondence.

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<sup>1</sup>See [11] for an earlier discussion of higher symmetries of the fundamental S-matrix.

The approach by Lüscher is closely related to the thermodynamic Bethe ansatz (TBA) [41]. In the TBA one deals with finite size effects by defining a mirror model [23]. Finite size effects in the original theory correspond to finite temperature effects in the infinite volume for the mirror theory. Here again, one needs to include all (physical) bound states. One of the advantages of this method is that one can still use the asymptotic Bethe ansatz. So far, this has only been carried out for the fundamental multiplets [8, 12, 13].

In other words, knowledge of the bound states, their S-matrices and the corresponding Bethe ansätze is crucial for a complete understanding of finite size effects. Because of the relation between Yangian symmetry and the bound state S-matrices, one might suspect that the Bethe equations are also closely related to Yangian symmetry. This indeed appears to be the case as we will explain in this note. The Bethe equations that are derived can be used to study bound states of the mirror model by analytic continuation [23].

The aim of this paper is to provide a rigorous derivation of the Bethe Ansatz equations for the bound states, by diagonalizing the multi-particle bound state S-matrices using Yangian symmetry as a tool. The explicit bound state number dependence in the Bethe equations appears through the parameters  $x^\pm$  and the dressing phase only. The Bethe equations emerging in this way, were found, a posteriori, to follow from a fusion procedure. This justifies the fusion procedure at the level of the Bethe ansatz equations.

The paper is organized as follows. In the first section we recall the basic formulas of the Yangian of the centrally extended  $\mathfrak{su}(2|2)$  algebra in the superspace formalism. Subsequently, we will discuss the Bethe ansatz for fundamental representations. Finally, the Bethe ansatz will be reformulated in terms of coproducts of the Yangian and the bound state Bethe equations will be derived.

## 2. Symmetry algebra, coproducts and S-matrices

In this paper the full (Yangian) symmetry algebra of the S-matrix is important. In this section we will give a brief overview of the Yangian symmetry of the bound state S-matrices. For more details see e.g [32] and references therein.

### 2.1 The algebra in superspace

We will first discuss centrally extended  $\mathfrak{su}(2|2)$ . This algebra is the symmetry algebra of the  $\text{AdS}_5 \times S^5$  superstring and it is also the symmetry algebra of the spin chain connected to  $\mathcal{N} = 4$  SYM. The algebra has bosonic generators  $\mathbb{R}, \mathbb{L}$ , supersymmetry generators  $\mathbb{Q}, \mathbb{G}$  and central charges  $\mathbb{H}, \mathbb{C}, \mathbb{C}^\dagger$ . The non-trivial commutation relations between the generators are given by

$$\begin{aligned}
 [\mathbb{L}_a^b, \mathbb{J}_c] &= \delta_c^b \mathbb{J}_a - \frac{1}{2} \delta_a^b \mathbb{J}_c & [\mathbb{R}_\alpha^\beta, \mathbb{J}_\gamma] &= \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2} \delta_\alpha^\beta \mathbb{J}_\gamma \\
 [\mathbb{L}_a^b, \mathbb{J}^c] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2} \delta_a^b \mathbb{J}^c & [\mathbb{R}_\alpha^\beta, \mathbb{J}^\gamma] &= -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2} \delta_\alpha^\beta \mathbb{J}^\gamma \\
 \{\mathbb{Q}_\alpha^a, \mathbb{Q}_\beta^b\} &= \epsilon_{\alpha\beta} \epsilon^{ab} \mathbb{C} & \{\mathbb{G}_a^\alpha, \mathbb{G}_b^\beta\} &= \epsilon^{\alpha\beta} \epsilon_{ab} \mathbb{C}^\dagger \\
 \{\mathbb{Q}_\alpha^a, \mathbb{G}_b^\beta\} &= \delta_b^a \mathbb{R}_\alpha^\beta + \delta_\alpha^\beta \mathbb{L}_b^a + \frac{1}{2} \delta_b^a \delta_\alpha^\beta \mathbb{H}. & & (2.1)
 \end{aligned}$$

The eigenvalues of the central charges are denoted by  $H, C, C^\dagger$ . The charge  $H$  is Hermitian and the charges  $C, C^\dagger$  and the generators  $\mathbb{Q}, \mathbb{G}$  are conjugate to each other.

For computational purposes, it proves worthwhile to consider representations of the algebra in the superspace formalism. Consider the vector space of analytic functions of two bosonic variables  $w_{1,2}$  and two fermionic variables  $\theta_{3,4}$ . Since we are dealing with analytic functions we can expand any such function  $\Phi(w, \theta)$ :

$$\begin{aligned} \Phi(w, \theta) &= \sum_{\ell=0}^{\infty} \Phi_\ell(w, \theta), \\ \Phi_\ell &= \phi^{a_1 \dots a_\ell} w_{a_1} \dots w_{a_\ell} + \phi^{a_1 \dots a_{\ell-1} \alpha} w_{a_1} \dots w_{a_{\ell-1}} \theta_\alpha \\ &\quad + \phi^{a_1 \dots a_{\ell-2} \alpha \beta} w_{a_1} \dots w_{a_{\ell-2}} \theta_\alpha \theta_\beta. \end{aligned} \quad (2.2)$$

The representation that describes  $\ell$ -particle bound states of the light-cone string theory on  $\text{AdS}_5 \times \text{S}^5$  of centrally extended  $\mathfrak{su}(2|2)$  is  $4\ell$  dimensional. It is realized on a graded vector space with basis  $|e_{a_1 \dots a_\ell}\rangle, |e_{a_1 \dots a_{\ell-1} \alpha}\rangle, |e_{a_1 \dots a_{\ell-2} \alpha \beta}\rangle$ , where  $a_i$  are bosonic indices and  $\alpha, \beta$  are fermionic indices and each of the basis vectors is totally symmetric in the bosonic indices and anti-symmetric in the fermionic indices [24, 25, 27]. In terms of the above analytic functions, the basis vectors of the totally symmetric representation can evidently be identified  $|e_{a_1 \dots a_\ell}\rangle \leftrightarrow w_{a_1} \dots w_{a_\ell}, |e_{a_1 \dots a_{\ell-1} \alpha}\rangle \leftrightarrow w_{a_1} \dots w_{a_{\ell-1}} \theta_\alpha$  and  $|e_{a_1 \dots a_{\ell-2} \alpha \beta}\rangle \leftrightarrow w_{a_1} \dots w_{a_{\ell-2}} \theta_\alpha \theta_\beta$  respectively. In other words, we find the atypical totally symmetric representation describing  $\ell$ -particle bound states when we restrict to terms  $\Phi_\ell$ . For later convenience, we introduce the notation  $W_{1^i 2^j 3^k 4^l}^{(m)} \equiv (w_1^{(m)})^i (w_2^{(m)})^j (\theta_3^{(m)})^k (\theta_4^{(m)})^l$ , where  $(m)$  denotes different copies of the representation. Clearly, for an  $\ell$ -particle bound state representation, this means that  $i + j + k + l = \ell$  and  $k, l = 0, 1$ .

In this representation the algebra generators can be written in differential operator form in the following way

$$\begin{aligned} \mathbb{L}_a^b &= w_a \frac{\partial}{\partial w_b} - \frac{1}{2} \delta_a^b w_c \frac{\partial}{\partial w_c}, & \mathbb{R}_\alpha^\beta &= \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2} \delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial \theta_\gamma}, \\ \mathbb{Q}_\alpha^a &= a \theta_\alpha \frac{\partial}{\partial w_a} + b \epsilon^{ab} \epsilon_{\alpha\beta} w_b \frac{\partial}{\partial \theta_\beta}, & \mathbb{G}_a^\alpha &= d w_a \frac{\partial}{\partial \theta_\alpha} + c \epsilon_{ab} \epsilon^{\alpha\beta} \theta_\beta \frac{\partial}{\partial w_b} \end{aligned} \quad (2.3)$$

and the central charges are

$$\begin{aligned} \mathbb{C} &= ab \left( w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right) & \mathbb{C}^\dagger &= cd \left( w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right) \\ \mathbb{H} &= (ad + bc) \left( w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right). \end{aligned} \quad (2.4)$$

To form a representation, the parameters  $a, b, c, d$  satisfy the condition  $ad - bc = 1$ . The central charges become  $\ell$  dependent:

$$H = \ell(ad + bc), \quad C = \ell ab, \quad C^\dagger = \ell cd. \quad (2.5)$$

The parameters  $a, b, c, d$  can be expressed in terms of the particle momentum  $p$  and the coupling  $g$ :

$$\begin{aligned} a &= \sqrt{\frac{g}{2\ell}}\eta & b &= \sqrt{\frac{g}{2\ell}}\frac{i\zeta}{\eta}\left(\frac{x^+}{x^-} - 1\right) \\ c &= -\sqrt{\frac{g}{2\ell}}\frac{\eta}{\zeta x^+} & d &= \sqrt{\frac{g}{2\ell}}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right), \end{aligned} \quad (2.6)$$

where the parameters  $x^\pm$  satisfy

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i\ell}{g}, \quad \frac{x^+}{x^-} = e^{ip} \quad (2.7)$$

and the parameters  $\eta$  are given by

$$\eta = e^{i\xi}\eta(p), \quad \eta(p) = e^{\frac{i}{4}p}\sqrt{ix^- - ix^+}, \quad \zeta = e^{2i\xi}. \quad (2.8)$$

The fundamental representation, which is used in the derivation of the S-matrix scattering fundamental multiplets [8, 11] is obtained by taking  $\ell = 1$ .

## 2.2 Yangians, coproducts and the S-matrix

The double Yangian  $DY(\mathfrak{g})$  of a (simple) Lie algebra  $\mathfrak{g}$  is a deformation of the universal enveloping algebra  $U(\mathfrak{g}[u, u^{-1}])$  of the loop algebra  $\mathfrak{g}[u, u^{-1}]$ . The Yangian is generated by level  $n$  generators  $\mathbb{J}_n^A$ ,  $n \in \mathbb{Z}$  that satisfy the commutation relations

$$[\mathbb{J}_m^A, \mathbb{J}_n^B] = F_C^{AB} \mathbb{J}_{m+n}^C + \mathcal{O}(\hbar), \quad (2.9)$$

where  $F_C^{AB}$  are the structure constants of  $\mathfrak{g}$ . The level-0 generators  $\mathbb{J}_0^A$  span the Lie-algebra. The  $\mathfrak{su}(2|2)$  Yangian can be supplied with a coproduct in the following way [31, 42, 43]

$$\begin{aligned} \Delta(\mathbb{J}_n^A) &= \mathbb{J}_n^A \otimes 1 + \mathcal{U}^{[A]} \otimes \mathbb{J}_n^A + \frac{\hbar}{2} \sum_{m=0}^{n-1} F_{BC}^A \mathbb{J}_{n-1-m}^B \mathcal{U}^{[C]} \otimes \mathbb{J}_m^C + \mathcal{O}(\hbar^2) \\ \Delta(\mathcal{U}) &= \mathcal{U} \otimes \mathcal{U}, \end{aligned} \quad (2.10)$$

where  $\mathcal{U}$  comprises a so-called braiding factor.

An important representation of a Yangian is the evaluation representation. This representation consists of states  $|u\rangle$ , with action  $\mathbb{J}_n^A|u\rangle = u^n \mathbb{J}_0^A|u\rangle$ . In this representation the coproduct structure is fixed in terms of the coproducts of  $\mathbb{J}_0, \mathbb{J}_1$ . For the remainder of this paper we will work in this representation and identify  $\mathbb{J}_1 \equiv \hat{\mathbb{J}} = \frac{g}{2i}u\mathbb{J}$  for the  $\mathfrak{su}(2|2)$  Yangian. One also finds that  $u$  is actually dependent on the parameters  $x^\pm$  via  $u_j = x_j^+ + \frac{1}{x_j^+} - \frac{i\ell_j}{g}$ .

The S-matrix is a map between the following representations:

$$\mathbb{S} : \mathcal{V}_{\ell_1}(p_1, e^{ip_2}) \otimes \mathcal{V}_{\ell_2}(p_2, 1) \longrightarrow \mathcal{V}_{\ell_1}(p_1, 1) \otimes \mathcal{V}_{\ell_2}(p_2, e^{ip_1}), \quad (2.11)$$

where  $\mathcal{V}_{\ell_i}(p_i, e^{2i\xi})$  is the  $\ell_i$ -bound state representation with parameters  $a_i, b_i, c_i, d_i$  with the explicit choice of  $\zeta = e^{2i\xi}$ . This specific choice removes the braid factor  $\mathcal{U}$  from the discussion [11].

The bound state S-matrices are now fixed, up to a overall phase, by requiring invariance under the coproducts of the (Yangian) symmetry generators

$$\begin{aligned}\mathbb{S} \Delta(\mathbb{J}^A) &= \Delta^{op}(\mathbb{J}^A) \mathbb{S} \\ \mathbb{S} \Delta(\hat{\mathbb{J}}^A) &= \Delta^{op}(\hat{\mathbb{J}}^A) \mathbb{S},\end{aligned}\tag{2.12}$$

where  $\Delta^{op} = P\Delta$  with  $P$  the graded permutation.

For completeness and future reference, we will give the explicit formulas of the different coproducts and of the parameters  $a_i, b_i, c_i, d_i$ . First, the  $\mathfrak{su}(2|2)$  operators:

$$\Delta(\mathbb{J}_0^A) = \mathbb{J}_{1;0}^A + \mathbb{J}_{2;0}^A.\tag{2.13}$$

The coproducts of the Yangian generators are

$$\begin{aligned}\Delta(\hat{\mathbb{L}}^a_b) &= \hat{\mathbb{L}}_{1;b}^a + \hat{\mathbb{L}}_{2;b}^a + \frac{1}{2}\mathbb{L}_{1;b}^c\mathbb{L}_{2;c}^a - \frac{1}{2}\mathbb{L}_{1;c}^a\mathbb{L}_{2;b}^c - \frac{1}{2}\mathbb{G}_{1;b}^\gamma\mathbb{Q}_{2;\gamma}^a - \frac{1}{2}\mathbb{Q}_{1;\gamma}^a\mathbb{G}_{2;b}^\gamma \\ &\quad + \frac{1}{4}\delta_b^a\mathbb{G}_{1;c}^\gamma\mathbb{Q}_{2;\gamma}^c + \frac{1}{4}\delta_b^a\mathbb{Q}_{1;\gamma}^c\mathbb{G}_{2;c}^\gamma \\ \Delta(\hat{\mathbb{R}}^\alpha_\beta) &= \hat{\mathbb{R}}_{1;\beta}^\alpha + \hat{\mathbb{R}}_{2;\beta}^\alpha - \frac{1}{2}\mathbb{R}_{1;\beta}^\gamma\mathbb{R}_{2;\gamma}^\alpha + \frac{1}{2}\mathbb{R}_{1;\gamma}^\alpha\mathbb{R}_{2;\beta}^\gamma + \frac{1}{2}\mathbb{G}_{1;c}^\alpha\mathbb{Q}_{2;\beta}^c + \frac{1}{2}\mathbb{Q}_{1;\beta}^c\mathbb{G}_{2;c}^\alpha \\ &\quad - \frac{1}{4}\delta_\beta^\alpha\mathbb{G}_{1;c}^\gamma\mathbb{Q}_{2;\gamma}^c - \frac{1}{4}\delta_\beta^\alpha\mathbb{Q}_{1;\gamma}^c\mathbb{G}_{2;c}^\gamma \\ \Delta(\hat{\mathbb{Q}}^a_\beta) &= \hat{\mathbb{Q}}_{1;\beta}^a + \hat{\mathbb{Q}}_{2;\beta}^a - \frac{1}{2}\mathbb{R}_{1;\beta}^\gamma\mathbb{Q}_{2;\gamma}^a + \frac{1}{2}\mathbb{Q}_{1;\gamma}^a\mathbb{R}_{2;\beta}^\gamma - \frac{1}{2}\mathbb{L}_{1;c}^a\mathbb{Q}_{2;\beta}^c + \frac{1}{2}\mathbb{Q}_{1;\beta}^c\mathbb{L}_{2;c}^a \\ &\quad - \frac{1}{4}\mathbb{H}_1\mathbb{Q}_{2;\beta}^a + \frac{1}{4}\mathbb{Q}_{1;\beta}^a\mathbb{H}_2 + \frac{1}{2}\epsilon_{\beta\gamma}\epsilon^{ad}\mathbb{C}_1\mathbb{G}_{2;d}^\gamma - \frac{1}{2}\epsilon_{\beta\gamma}\epsilon^{ad}\mathbb{G}_{1;d}^\gamma\mathbb{C}_2 \\ \Delta(\hat{\mathbb{G}}^\alpha_b) &= \hat{\mathbb{G}}_{1;b}^\alpha + \hat{\mathbb{G}}_{2;b}^\alpha + \frac{1}{2}\mathbb{L}_{1;b}^c\mathbb{G}_{2;c}^\alpha - \frac{1}{2}\mathbb{G}_{1;c}^\alpha\mathbb{L}_{2;b}^c + \frac{1}{2}\mathbb{R}_{1;\gamma}^\alpha\mathbb{G}_{2;b}^\gamma - \frac{1}{2}\mathbb{G}_{1;b}^\gamma\mathbb{R}_{2;\gamma}^\alpha \\ &\quad + \frac{1}{4}\mathbb{H}_1\mathbb{G}_{2;b}^\alpha - \frac{1}{4}\mathbb{G}_{1;b}^\alpha\mathbb{H}_2 - \frac{1}{2}\epsilon_{bc}\epsilon^{\alpha\gamma}\mathbb{C}_1^\dagger\mathbb{Q}_{2;\gamma}^c + \frac{1}{2}\epsilon_{bc}\epsilon^{\alpha\gamma}\mathbb{Q}_{1;\gamma}^c\mathbb{C}_2^\dagger\end{aligned}\tag{2.14}$$

and for the central charges

$$\begin{aligned}\Delta(\hat{\mathbb{H}}) &= \hat{\mathbb{H}}_1 + \hat{\mathbb{H}}_2 + \mathbb{C}_1\mathbb{C}_2^\dagger - \mathbb{C}_1^\dagger\mathbb{C}_2 \\ \Delta(\hat{\mathbb{C}}) &= \hat{\mathbb{C}}_1 + \hat{\mathbb{C}}_2 + \frac{1}{2}\mathbb{H}_1\mathbb{C}_2 - \frac{1}{2}\mathbb{C}_1\mathbb{H}_2 \\ \Delta(\hat{\mathbb{C}}^\dagger) &= \hat{\mathbb{C}}_1^\dagger + \hat{\mathbb{C}}_2^\dagger + \frac{1}{2}\mathbb{H}_1\mathbb{C}_2^\dagger - \frac{1}{2}\mathbb{C}_1^\dagger\mathbb{H}_2.\end{aligned}\tag{2.15}$$

The product is ordered, e.g.  $\mathbb{Q}_1\mathbb{Q}_2$  means first apply  $\mathbb{Q}_2$ , then  $\mathbb{Q}_1$  (as differential operators). Finally, the coefficients used in  $\Delta$  are given by:

$$\begin{aligned}a_1 &= \sqrt{\frac{g}{2\ell_1}}\eta_1 & b_1 &= -ie^{ip_2}\sqrt{\frac{g}{2\ell_1}}\frac{1}{\eta_1}\left(\frac{x_1^+}{x_1^-} - 1\right) \\ c_1 &= -e^{-ip_2}\sqrt{\frac{g}{2\ell_1}}\frac{\eta_1}{x_1^+} & d_1 &= i\sqrt{\frac{g}{2\ell_1}}\frac{x_1^+}{\eta_1}\left(\frac{x_1^-}{x_1^+} - 1\right) \\ \eta_1 &= e^{i\frac{p_1}{4}}e^{i\frac{p_2}{2}}\sqrt{ix_1^- - ix_1^+}\end{aligned}\tag{2.16}$$



$$\begin{aligned}
 a_2 &= \sqrt{\frac{g}{2\ell_2}} \eta_2 & b_2 &= -i \sqrt{\frac{g}{2\ell_2}} \frac{1}{\eta_2} \left( \frac{x_2^+}{x_2^-} - 1 \right) \\
 c_2 &= -\sqrt{\frac{g}{2\ell_2}} \frac{\eta_2}{x_2^+} & d_2 &= i \sqrt{\frac{g}{2\ell_2}} \frac{x_2^+}{i\eta_2} \left( \frac{x_2^-}{x_2^+} - 1 \right) \\
 \eta_2 &= e^{i\frac{p_2}{4}} \sqrt{ix_2^- - ix_2^+}
 \end{aligned} \tag{2.17}$$

The coefficients in  $\Delta^{op}$  are given by:

$$\begin{aligned}
 a_1^{op} &= \sqrt{\frac{g}{2\ell_1}} \eta_1^{op} & b_1^{op} &= -i \sqrt{\frac{g}{2\ell_1}} \frac{1}{\eta_1^{op}} \left( \frac{x_1^+}{x_1^-} - 1 \right) \\
 c_1^{op} &= -\sqrt{\frac{g}{2\ell_1}} \frac{\eta_1^{op}}{x_1^+} & d_1^{op} &= i \sqrt{\frac{g}{2\ell_1}} \frac{x_1^+}{i\eta_1^{op}} \left( \frac{x_1^-}{x_1^+} - 1 \right) \\
 \eta_1^{op} &= e^{i\frac{p_1}{4}} \sqrt{ix_1^- - ix_1^+} \\
 a_2^{op} &= \sqrt{\frac{g}{2\ell_2}} \eta_2^{op} & b_2^{op} &= -ie^{ip_1} \sqrt{\frac{g}{2\ell_2}} \frac{1}{\eta_2^{op}} \left( \frac{x_2^+}{x_2^-} - 1 \right) \\
 c_2^{op} &= -e^{-ip_1} \sqrt{\frac{g}{2\ell_2}} \frac{\eta_2^{op}}{x_2^+} & d_2^{op} &= i \sqrt{\frac{g}{2\ell_2}} \frac{x_2^+}{\eta_2^{op}} \left( \frac{x_2^-}{x_2^+} - 1 \right) \\
 \eta_2^{op} &= e^{i\frac{p_2}{4}} e^{i\frac{p_1}{2}} \sqrt{ix_2^- - ix_2^+}
 \end{aligned} \tag{2.18}$$

The non-trivial braiding factors are all hidden in the parameters of the four representations involved.

### 3. The $\mathfrak{su}(2|2)$ coordinate Bethe Ansatz

In this section we will briefly discuss the coordinate Bethe Ansatz for the  $\mathfrak{su}(2|2)$  string S-matrix as done in [8, 13].

#### 3.1 Formalism

Consider  $K^1$  excitations with momentum  $p_1, \dots, p_{K^1}$ . We divide our space into regions  $\mathcal{P}|\mathcal{Q}$ , with  $\mathcal{P}, \mathcal{Q}$  permutations. In the sector where the particle coordinates are ordered as  $x_{\mathcal{Q}^1} < \dots < x_{\mathcal{Q}^{K^1}}$  the ansatz for the wave function is given by

$$|p_1, \dots, p_{K^1}\rangle = \sum_{\mathcal{P}} \int dx \left\{ A_{a_1 \dots a_{K^1}}^{\mathcal{P}|\mathcal{Q}} e^{ip_{\mathcal{P}_i} x_{\mathcal{Q}_i}} \right\} \phi^{a_1}(x_1) \dots \phi^{a_{K^1}}(x_{K^1}) \tag{3.1}$$

The indices  $a_i$  denote the type of the  $i$ th particle and  $\phi^{a_i}(x_i)$  is a creation operator that creates a particle of type  $a_i$  at position  $x_i$ . In general when the system contains bound states, the indices  $a_i$  also run over bound states. The different sectors are related to each other by permutations and S-matrices. To be more precise

$$A^{\mathcal{P}|\mathcal{Q}} = \mathbb{S}_{i,j} A^{\mathcal{P}'|\mathcal{Q}'}, \tag{3.2}$$

where  $P'|Q'$  are the permutations obtained from  $P|Q$  by interchanging the neighboring  $i$ th and  $j$ th particles. To make this transformation property more explicit we will write

$$A_{a_1 \dots a_K}^{\mathcal{P}|Q} \equiv A_{a_1 \dots a_K}^{\mathcal{P}'|Q'} w_{a_1} \dots w_{a_K}, \quad (3.3)$$

where we do not sum over repeated indices. Periodicity of the total wave function is then formulated in the following Bethe equations

$$\mathbb{S}_k A^{\mathcal{P}|Q} := \mathbb{S}_{kk-1} \dots \mathbb{S}_{kK^1} \mathbb{S}_{k1} \dots \mathbb{S}_{kk+1} A^{\mathcal{P}|Q} = e^{ip_k L} A^{\mathcal{P}|Q}. \quad (3.4)$$

The Bethe ansatz now consists of making an ansatz for the coefficients  $A^{\mathcal{P}|Q}$  in such a way that they solve (3.2). This ansatz is then plugged in the above formula and the explicit Bethe equations can be read off.

Obviously, since any scattering process reduces to a product of two body scattering processes, we only need to consider two sites of the wave function. In what follows we will solve for the coefficients  $A_{i_1 \dots i_{K^1}}^{\mathcal{P}|Q}$ .

### 3.2 Solving the coefficients

We will now solve the coefficients for the  $\mathfrak{su}(2|2)$  S-matrix for fundamental representations. In order to make the derivation more transparent, we explicitly identify the coefficients  $A^{\mathcal{P}|Q}$  with states  $|\cdot\rangle \in \mathcal{V}_1(p_1) \otimes \dots \otimes \mathcal{V}_1(p_{K^1})$ . From now on we will omit the explicit mentioning of the sector  $\mathcal{P}|Q$  we are in. The different sectors are related via relabelling of momenta and interchanging of particle positions and hence the coefficients should be well-defined in the sense that they should respect this. From (3.2), we see that in order to be well-defined, we must have that

$$\mathbb{S}|\phi\rangle = |\phi\rangle_\pi, \quad (3.5)$$

where  $|\phi\rangle_\pi$  is the coefficient as constructed in the sector  $\mathcal{P}'|Q'$ .

Let us first define the vacuum:

$$|0\rangle = W_1^{(1)} \dots W_1^{(K)}. \quad (3.6)$$

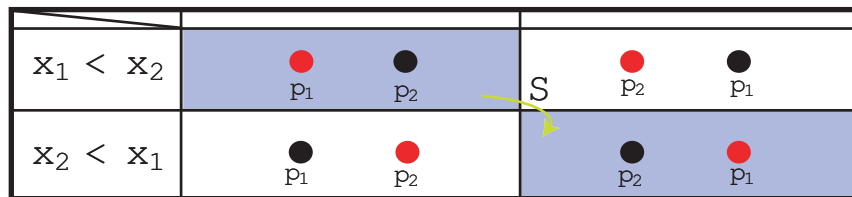
One can also take other vacua but this leads to equivalent equations. If we, for the moment, consider the undressed S-matrix, we find that

$$\mathbb{S}|0\rangle = |0\rangle. \quad (3.7)$$

In the light of (3.5), we indeed find that this vacuum corresponds to a vacuum in all sectors  $\mathcal{P}|Q$  and is well defined.

The complete dressing phase  $S_0(p_i, p_j)$  can be easily incorporated by a rescaling

$$|0\rangle = \frac{W_1^{(1)} \dots W_1^{(K)}}{\sqrt{\prod_{i \neq j} S_0(p_i, p_j)}}. \quad (3.8)$$



**Figure 1:** Example of the construction of the coefficients in the different sectors. The black dots correspond to the bosons of the vacuum,  $W_1$ , and the red dots correspond to the inserted fermions. The construction is well-defined if, by acting on the coefficient in the upper left quadrant by the S-matrix, one obtains the coefficient in the lower right quadrant.

This works because  $S_0(p_j, p_i) = S_0(p_j, p_i)^{-1}$ . To avoid making the formulas too cumbersome, we will restrict to the undressed S-matrix in the remainder of the paper. We will, of course, include the dressing phase in the full Bethe equations.

The next thing to consider is the case where we have a fermion in this vacuum (the case in which the other boson is inserted is treated later on). We make an ansatz of the following form:

$$|\alpha\rangle := \sum_i \Psi_i(y) W_1^{(1)} \dots W_\alpha^{(i)} \dots W_1^{(K)}, \quad \Psi_k(y) = f(y, p_k) \prod_{l < k} S^{\text{II,I}}(y, p_l). \quad (3.9)$$

We will denote  $S(y, p) \equiv S^{\text{II,I}}(y, p)$ . We must check whether this construction is well defined in the sense that it respects (3.5). How this works is schematically depicted in figure 1.

By the factorization property of the S-matrix, it suffices to restrict to a two particle state and a two particle S-matrix. For this case (3.5) gives the following equations

$$\begin{aligned} \frac{e^{i\frac{p_1}{2}} \eta(p_2) x_1^+ - x_1^-}{e^{i\frac{p_2}{2}} \eta(p_1) x_1^+ - x_2^-} f(y, p_1) + e^{i\frac{p_1}{2}} \frac{x_1^- - x_2^-}{x_1^+ - x_2^-} f(y, p_2) S(y, p_1) &= f(y, p_2) \\ e^{-i\frac{p_2}{2}} \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} f(y, p_1) + \frac{\eta(p_1) x_2^+ - x_2^-}{\eta(p_2) x_1^+ - x_2^-} f(y, p_2) S(y, p_1) &= f(y, p_1) S(y, p_2). \end{aligned} \quad (3.10)$$

These equations can be solved explicitly and the solution is given by:

$$\begin{aligned} f(y, p_k) &= \eta(p_k) \sqrt{\frac{x_k^-}{x_k^+}} \frac{y}{y - x_k^-} \sqrt{\frac{g\ell_k}{2}} \\ S(y, p_k) &= \sqrt{\frac{x_k^- y - x_k^+}{x_k^+ y - x_k^-}}, \end{aligned} \quad (3.11)$$

where  $y$  enters as a integration constant. With a modest amount of foresight, we choose the overall normalization of  $f$  to be dependent on the bound state number  $\ell_k$ .

Actually this solution is the unique solution of (3.5) for one fermion in the vacuum up to overall normalization (and hence the only well-defined coefficient for this case). This can easily be seen from (3.10). The general wave function is of the form

$$|\alpha\rangle := A_1 W_\alpha^{(1)} \dots W_1^{(K)} + A_2 W_1^{(1)} W_\alpha^{(2)} \dots W_1^{(K)} + \dots \quad (3.12)$$

Since we are only interested in solving (3.4), the normalization of the state is irrelevant. Let us therefore pick a convenient normalization,  $A_1 \equiv f(p_1)$ , then by the first equation of (3.10) one finds  $A_2$  in terms of  $A_1$ . This indeed fixes  $A_2$  as  $f(p_2)S(p_1)$  and by induction the rest of the coefficients also follow. Finally, let us stress that by fixing the normalization, one automatically finds from the S-matrix that the other coefficients of the wave function are of a factorized form.

The problem becomes more involved upon inserting two excitations. In this case, it appears that the ansatz for the coefficient can be written as

$$|\alpha\beta\rangle = |\alpha\beta\rangle_{y_1 y_2} + \mathbb{S}^{\text{II}} |\alpha\beta\rangle_{y_1 y_2}, \quad (3.13)$$

where we introduce a new S-matrix  $\mathbb{S}^{\text{II}}$

$$\mathbb{S}^{\text{II}} |\alpha\beta\rangle_{y_1 y_2} = M(y_1, y_2) |\alpha\beta\rangle_{y_2 y_1} + N(y_1, y_2) |\beta\alpha\rangle_{y_2 y_1}. \quad (3.14)$$

Explicitly, it is given by

$$\begin{aligned} |\alpha\beta\rangle = & \sum_{k < l} \Psi_k(y_1) \Psi_l(y_2) W_1^{(1)} \dots W_\alpha^{(k)} \dots W_\beta^{(l)} \dots W_1^{(K)} + \\ & + \mathbb{S}^{\text{II}} \sum_{k < l} \Psi_k(y_1) \Psi_l(y_2) W_1^{(1)} \dots W_\alpha^{(k)} \dots W_\beta^{(l)} \dots W_1^{(K)} + \\ & + \epsilon^{\alpha\beta} \sum_k \Psi_k(y_1) \Psi_k(y_2) h(y_1, y_2, p_k) W_1^{(1)} \dots W_2^{(k)} \dots W_1^{(K)} \end{aligned} \quad (3.15)$$

When we restrict to just two sites, the wave function splits into the sum of a wave function with just one fermion. By construction, this piece is exactly as described above. The other piece contains two excitations and is given by

$$\begin{aligned} |\alpha\beta\rangle = & \{f(y_1, p_1) f(y_2, p_2) S(y_2, p_1) + M f(y_2, p_1) f(y_1, p_2) S(y_1, p_1)\} W_\alpha^{(1)} W_\beta^{(2)} \\ & + N f(y_2, p_1) f(y_1, p_2) S(y_1, p_1) W_\beta^{(1)} W_\alpha^{(2)} \\ & + \epsilon^{\alpha\beta} h(y_1, y_2, p_1) f(y_2, p_1) f(y_1, p_1) W_2^{(1)} W_1^{(2)} \\ & + \epsilon^{\alpha\beta} h(y_1, y_2, p_2) f(y_2, p_2) f(y_1, p_2) S(y_2, p_1) S(y_1, p_1) W_1^{(1)} W_2^{(2)} \end{aligned} \quad (3.16)$$

Plugging this into (3.5) again allows one to find the explicit (unique) solutions of the unknown functions:

$$\begin{aligned} M(y_1, y_2) &= \frac{2i/g}{y_1 + \frac{1}{y_1} - y_2 - \frac{1}{y_2} - \frac{2i}{g}} \\ N(y_1, y_2) &= -\frac{y_1 + \frac{1}{y_1} - y_2 - \frac{1}{y_2}}{y_1 + \frac{1}{y_1} - y_2 - \frac{1}{y_2} - \frac{2i}{g}} \\ h(y_1, y_2, p_k) &= \frac{i}{\ell_k \eta(p_k)^2} \frac{y_1 y_2 - x_k^+ x_k^-}{y_1 y_2} \frac{x_k^+ - x_k^-}{x_k^-} \frac{y_1 - y_2}{y_1 + \frac{1}{y_1} - y_2 - \frac{1}{y_2} - \frac{2i}{g}} \end{aligned} \quad (3.17)$$

In this process, we introduced a new S-matrix and we also insist on naturalness of the wave function with respect to this S-matrix,  $\mathbb{S}^{\text{II}}$ . One can repeat the above procedure to deal

with this. For details we refer to [13, 8]. In this process we are again lead to introduce new functions  $S^{\text{II,II}}, f^{(2)}, S^{\text{III,II}}, S^{\text{III,III}}$ . The result of this consideration is

$$\begin{aligned}
 S^{\text{II,II}} &= -M - N = 1 \\
 f^{(2)}(w, y_k) &= \frac{w - \frac{i}{g}}{w - v_k - \frac{i}{g}} \\
 S^{\text{III,II}}(w, y_k) &= \frac{w - v_k + \frac{i}{g}}{w - v_k - \frac{i}{g}} \\
 S^{\text{III,III}}(w_1, w_2) &= \frac{w_1 - w_2 - \frac{2i}{g}}{w_1 - w_2 + \frac{2i}{g}},
 \end{aligned} \tag{3.18}$$

The Bethe equations (3.4) are formulated in terms of the factors used in the ansatz in the following way

$$e^{iL_{A,k}} = \prod_{B=1}^{\text{III}} \prod_{l=1}^{K^B} S^{BA}(x_l^B, x_k^A), \quad (B, l) \neq (A, k) \tag{3.19}$$

where  $A, B$  denote the different levels and  $S^{AB}$  can be seen as the S-matrix describing scattering at different levels. Moreover,  $e^{iL_{1,k}} = e^{-iL_{p,k}}$  and  $e^{iL_{\text{II},k}} = e^{iL_{\text{III},k}} = 1$  are phases.

By putting all of this together one obtains the well-known Bethe equations describing the asymptotic spectrum of the  $\text{AdS}_5 \times \text{S}^5$  superstring:

$$\begin{aligned}
 e^{ip_k L} &= \prod_{l=1, l \neq k}^{K^{\text{I}}} \left[ S_0(p_k, p_l) \frac{x_k^+ - x_l^-}{x_k^- - x_l^+} \sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}} \right]^2 \prod_{\alpha=1}^2 \prod_{l=1}^{K^{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \\
 1 &= \prod_{l=1}^{K^{\text{I}}} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_k^-}{x_k^+}} \prod_{l=1}^{K^{\text{III}}} \frac{y_k^{(\alpha)} + \frac{1}{y_k} - w_l^{(\alpha)} + \frac{i}{g}}{y_k^{(\alpha)} + \frac{1}{y_k} - w_l^{(\alpha)} - \frac{i}{g}} \\
 1 &= \prod_{l=1}^{K^{(\alpha)}} \frac{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} + \frac{i}{g}}{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} - \frac{i}{g}} \prod_{l \neq k}^{K^{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}},
 \end{aligned} \tag{3.20}$$

where  $\alpha = 1, 2$  reflect the two copies of  $\mathfrak{su}(2|2)$  and  $S_0(p_k, p_l)$  is the overall phase of the S-matrix.

Notice that in this construction, the explicit form of the S-matrix is used. However, since not all bound state S-matrices are known, one must approach this problem in a different way if one wishes to find their Bethe equations.

#### 4. Bethe ansatz and yangian symmetry

In this section we will generalize the above construction to arbitrary bound states. We will do this by considering coproducts of (Yangian) symmetry generators. This formulation allows us to solve (3.5) without knowing the explicit form of the involved bound state S-matrix. This will lead to the Bethe equations for arbitrary configurations of bound states.

#### 4.1 Single excitations

We will again start by considering a single excitation in the vacuum

$$|0\rangle = W_{1^{\ell_1}}^{(1)} \cdots W_{1^{\ell_K}}^{(K)}. \quad (4.1)$$

As noted above, it suffices to restrict to two bound state representations. The natural generalization of a single excitation wave function is:

$$|\alpha\rangle := \sum_i \Psi_i(y) W_{1^{\ell_1}}^{(1)} \cdots W_{3^{1^{\ell_i-1}}}^{(i)} \cdots W_{1^{\ell_K}}^{(K)}, \quad \Psi_k(y) = f(y, p_k) \prod_{l < k} S^{\text{I,II}}(y, p_l), \quad (4.2)$$

Restricted to two sites, the wave function is of the form

$$|\alpha\rangle = f(p_1) W_{\alpha 1^{\ell_1-1}}^{(1)} W_{1^{\ell_2}}^{(2)} + f(p_2) S(p_1) W_{1^{\ell_1}}^{(1)} W_{\alpha 1^{\ell_2-1}}^{(2)}, \quad (4.3)$$

where we again have chosen a particular normalization. The remarkable fact is that one can write this as:

$$\tilde{\Delta} Q_\alpha^1 |0\rangle := \left( K_0(p_1, p_2) \Delta Q_\alpha^1 + K_1(p_1, p_2) \Delta \hat{Q}_\alpha^1 \right) |0\rangle, \quad (4.4)$$

with

$$K_0 = -\sqrt{\frac{2}{g}} \frac{x_2^- \{x_1^- x_2^- (x_1^+ x_2^+ - 1) - x_1^+ x_2^+ (1 + x_1^- [x_1^- + x_1^+ + x_2^+])\}}{(x_2^- - x_1^+) (2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+)} \times \\ \times \left[ \frac{f(p_2) S(p_1)}{\sqrt{\ell_2} \eta(p_2)} - \frac{e^{-i\frac{p_2}{2}} f(p_1)}{\sqrt{\ell_1} \eta(p_1)} \right] + \frac{e^{-i\frac{p_2}{2}} f(p_1)}{\sqrt{\ell_1} \eta(p_1)} \quad (4.5)$$

$$K_1 = \frac{4i\sqrt{2}}{g^{3/2}} \frac{x_1^- x_2^- x_1^+ x_2^+}{(x_2^- - x_1^+) (2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+)} \left[ \frac{f(p_2) S(p_1)}{\sqrt{\ell_2} \eta(p_2)} - \frac{e^{-i\frac{p_2}{2}} f(p_1)}{\sqrt{\ell_1} \eta(p_1)} \right]$$

For the moment let us keep  $f, S$  arbitrary. The invariance of the S-matrix under Yangian symmetry means that

$$\mathbb{S} \Delta Q_\alpha^1 = \Delta^{op} Q_\alpha^1 \mathbb{S}, \quad \mathbb{S} \Delta \hat{Q}_\alpha^1 = \Delta^{op} \hat{Q}_\alpha^1 \mathbb{S}. \quad (4.6)$$

In other words, we find:

$$\begin{aligned} \mathbb{S} |\alpha\rangle &= \mathbb{S} \left( K_0(p_1, p_2) \Delta Q_\alpha^1 + K_1(p_1, p_2) \Delta \hat{Q}_\alpha^1 \right) |0\rangle \\ &= \left( K_0(p_1, p_2) \Delta^{op} Q_\alpha^1 + K_1(p_1, p_2) \Delta^{op} \hat{Q}_\alpha^1 \right) \mathbb{S} |0\rangle \end{aligned} \quad (4.7)$$

$$= \left( K_0(p_1, p_2) \Delta^{op} Q_\alpha^1 + K_1(p_1, p_2) \Delta^{op} \hat{Q}_\alpha^1 \right) |0\rangle, \quad (4.8)$$

since  $\mathbb{S} |0\rangle = |0\rangle$ . However, we also have

$$|\alpha\rangle_\pi = \left( K_0(p_2, p_1) \Delta^{op} Q_\alpha^1 + K_1(p_2, p_1) \Delta^{op} \hat{Q}_\alpha^1 \right) |0\rangle. \quad (4.9)$$

This means that (3.5) corresponds to requiring that  $K_0$  and  $K_1$  are symmetric under interchanging  $p_1 \leftrightarrow p_2$ . In other words, to find a well-defined coefficient, we have to solve

$$K_0(p_1, p_2) = K_0(p_2, p_1), \quad K_1(p_1, p_2) = K_1(p_2, p_1), \quad (4.10)$$

for the functions  $f$  and  $S$ .

It is straightforward to prove that (4.10) is equivalent to the equations:

$$\begin{aligned} Kf(p_1) + Gf(p_2)S(p_1) &= f(p_2) \\ Lf(p_1) + Hf(p_2)S(p_1) &= f(p_1)S(p_2), \end{aligned} \quad (4.11)$$

with

$$\begin{aligned} K &= \frac{e^{i\frac{p_1}{2}} \sqrt{\ell_2} \eta(p_2) x_1^+ - x_1^-}{e^{i\frac{p_2}{2}} \sqrt{\ell_1} \eta(p_1) x_1^+ - x_2^-} & G &= e^{i\frac{p_1}{2}} \frac{x_1^- - x_2^-}{x_1^+ - x_2^-} \\ L &= e^{-i\frac{p_2}{2}} \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} & H &= \frac{\sqrt{\ell_1} \eta(p_1) x_2^+ - x_2^-}{\sqrt{\ell_2} \eta(p_2) x_1^+ - x_2^-}. \end{aligned} \quad (4.12)$$

These equations are solved by the  $f, S$  found before, i.e. we again find (3.11) as unique solution. The discussion from the previous section holds also here. By fixing the normalization of the first term to be  $f(p_1)$ , one finds that the second term is factorized and fixed.

Moreover, notice that from this construction we can read off elements of the S-matrix. Namely, the coefficients that deal with the scattering of  $W_{\alpha a^{\ell_1-1}}^{(1)} W_{a^{\ell_2}}^{(2)}$  and  $W_{a^{\ell_1}}^{(1)} W_{\alpha a^{\ell_2-1}}^{(2)}$ . We compared the found coefficients with the explicit known S-matrices and we indeed find perfect agreement. For example, for  $\mathbb{S}^{BB}$  they coincide with  $a_9, a_{10}, a_{31}$  and  $a_{32}$ , cf. [27].

In conclusion, symmetry of the coefficients uniquely fixes the form of our wave function. We can now write the wave function, restricted to two sites, completely in terms of coproducts and as a consequence (3.5) is automatically satisfied. Finally, the explicit expressions for  $K_0, K_1$  are

$$\begin{aligned} K_0(p_1, p_2, y) &= \sqrt{\frac{x_1^-}{x_1^+}} \sqrt{\frac{x_2^-}{x_2^+}} \frac{y}{(y-x_1^-)(y-x_2^-)} \left[ y - \frac{x_1^- x_2^- x_1^+ x_2^+ (x_1^- + x_2^- + x_1^+ + x_2^+)}{2x_1^- x_2^- x_1^+ x_2^+ - x_1^- x_2^- - x_1^+ x_2^+} \right] \\ K_1(p_1, p_2, y) &= \frac{4i}{g} \sqrt{\frac{x_1^-}{x_1^+}} \sqrt{\frac{x_2^-}{x_2^+}} \frac{y}{(y-x_1^-)(y-x_2^-)} \left[ \frac{x_1^- x_2^- x_1^+ x_2^+}{2x_1^- x_2^- x_1^+ x_2^+ - x_1^- x_2^- - x_1^+ x_2^+} \right]. \end{aligned} \quad (4.13)$$

This consideration is valid for *any* bound state numbers and hence wave function (4.2) is valid for *any* bound state representations. In particular, all bound state representations share the same  $S^{\text{I,II}}$ , and hence that part of the Bethe equations remains the same.

## 4.2 Multiple excitations

When dealing with two excitations, one needs to introduce a level II S-matrix that deals with interchanging  $y_1$  and  $y_2$ .

**Fundamental representations.** Let us first restrict to fundamental representations and reformulate this in terms of coproducts. The wave function was of the form

$$|\alpha\beta\rangle = |\alpha\beta\rangle_{y_1 y_2} + \mathbb{S}^{\text{II}} |\alpha\beta\rangle_{y_1 y_2}, \quad (4.14)$$

where

$$\mathbb{S}^{\text{II}} |\alpha\beta\rangle_{y_1 y_2} = M(y_1, y_2) |\alpha\beta\rangle_{y_2 y_1} + N(y_1, y_2) |\beta\alpha\rangle_{y_2 y_1}. \quad (4.15)$$

The natural way to write this would be:

$$|\alpha\beta\rangle_{y_1 y_2} = \left\{ (\tilde{\Delta}_{y_1} \mathbb{Q}_\alpha^1)(\tilde{\Delta}_{y_2} \mathbb{Q}_\beta^1) + \epsilon_{\alpha\beta} \Delta'_{y_1, y_2} \mathbb{L}_2^1 \right\} |0\rangle, \quad (4.16)$$

with

$$\Delta'_{y_1, y_2} \mathbb{L}_2^1 := L_0(y_1, y_2, p_1, p_2) \Delta \mathbb{L}_2^1 + L_1(y_1, y_2, p_1, p_2) \Delta \hat{\mathbb{L}}_2^1 \quad (4.17)$$

By taking  $\alpha = \beta$ , one easily checks that the first part is indeed of the form  $(\tilde{\Delta}_{y_1} \mathbb{Q}_\alpha^1)(\tilde{\Delta}_{y_2} \mathbb{Q}_\beta^1)$ . Hence, we have to solve  $L_0, L_1$  such that our ansatz gives

$$\begin{aligned} & \{f(y_1, p_1)f(y_2, p_2)S(y_2, p_1) + Mf(y_2, p_1)f(y_1, p_2)S(y_1, p_1)\} W_\alpha^{(1)} W_\beta^{(2)} \\ & + Nf(y_2, p_1)f(y_1, p_2)S(y_1, p_1) W_\beta^{(1)} W_\alpha^{(2)} \\ & + \epsilon^{\alpha\beta} h(y_1, y_2, p_1) f(y_2, p_1) f(y_1, p_1) W_2^{(1)} W_1^{(2)} \\ & + \epsilon^{\alpha\beta} h(y_1, y_2, p_2) f(y_2, p_2) f(y_1, p_2) S(y_2, p_1) S(y_1, p_1) W_1^{(1)} W_2^{(2)}, \end{aligned} \quad (4.18)$$

where we keep the functions  $M, N, h$  arbitrary. This gives two equations for  $L_0$  and two equations for  $L_1$ . Hence, requiring symmetry under  $p_1 \leftrightarrow p_2$  will give us four equations which can be shown to be equivalent to the following set of equations:

$$\begin{aligned} \{f_{12} f_{21} S_{22} + M f_{22} f_{11} S_{12}\} &= \{f_{11} f_{22} S_{21} + M f_{21} f_{12} S_{11}\} \frac{D+E}{2} + N f_{21} f_{12} S_{11} \frac{D-E}{2} \\ &+ (-f_{11} f_{21} h_{121} + f_{12} f_{22} S_{11} S_{21} h_{122}) \frac{C}{2} \\ N f_{22} f_{11} S_{12} &= \{f_{11} f_{22} S_{21} + M f_{21} f_{12} S_{11}\} \frac{D-E}{2} + N f_{21} f_{12} S_{11} \frac{D+E}{2} \\ &- (-f_{11} f_{21} h_{121} + f_{12} f_{22} S_{11} S_{21} h_{122}) \frac{C}{2}. \\ f_{11} f_{21} S_{12} S_{22} h_{121} &= \{f_{11} f_{22} S_{21} + (M-N) f_{21} f_{12} S_{11}\} \frac{F}{2} \\ &+ f_{11} f_{21} h_{121} \frac{1-B}{2} + f_{12} f_{22} S_{11} S_{21} h_{122} \frac{1+B}{2} \\ f_{12} f_{22} h_{122} &= -\{f_{11} f_{22} S_{21} + (M-N) f_{21} f_{12} S_{11}\} \frac{F}{2} \\ &+ f_{11} f_{21} h_{121} \frac{1+B}{2} + f_{12} f_{22} S_{11} S_{21} h_{122} \frac{1-B}{2}, \end{aligned} \quad (4.19)$$

$$\begin{aligned} f_{12} f_{22} h_{122} &= -\{f_{11} f_{22} S_{21} + (M-N) f_{21} f_{12} S_{11}\} \frac{F}{2} \\ &+ f_{11} f_{21} h_{121} \frac{1+B}{2} + f_{12} f_{22} S_{11} S_{21} h_{122} \frac{1-B}{2}, \end{aligned} \quad (4.20)$$

where, for convenience, we introduced the short-hand notation  $f_{kl} := f(y_k, p_l)$ ,  $S_{kl} := S^{\text{II},1}(y_k, p_l)$ ,  $M := M(y_1, y_2)$ ,  $N := N(y_1, y_2)$  and  $h_{ijk} := h(y_i, y_j, p_k)$ . The coefficients  $B, C, D, E, F$  are given by

$$\begin{aligned} B &= \frac{2x_1^- x_2^- (x_2^+)^2 - (x_1^- x_2^- + 1)(x_2^- + x_1^+) x_2^+ + 2x_2^- x_1^+}{(1 - x_1^- x_2^-)(x_1^+ - x_2^-) x_2^+} \\ C &= 2i\eta(p_1)\eta(p_2) \frac{x_2^-}{x_2^+} \frac{e^{-\frac{ip_1}{2}} (x_2^+ - x_1^+)}{(1 - x_1^- x_2^-)(x_1^+ - x_2^-)} \\ D &= \frac{x_1^- - x_2^+}{x_2^- - x_1^+} \frac{e^{\frac{ip_1}{2}}}{e^{\frac{ip_2}{2}}} \end{aligned} \quad (4.21)$$



$$\begin{aligned}
 E &= \frac{e^{\frac{ip_1}{2}}}{e^{\frac{ip_2}{2}}} \frac{(x_1^-(x_2^-(x_1^- - 2x_1^+) + 1)x_1^+ + (x_1^+ + x_1^-(x_2^- x_1^+ - 2))x_2^+)}{(1 - x_1^- x_2^-)(x_1^+ - x_2^-)x_1^+} \\
 F &= 2i \frac{e^{-\frac{ip_1}{2}}}{\eta(p_1)\eta(p_2)} \frac{(x_1^+ - x_1^-)(x_2^+ - x_2^-)(x_2^+ - x_1^+)}{(1 - x_1^- x_2^-)(x_1^+ - x_2^-)}.
 \end{aligned}$$

It is readily seen that these expressions coincide with elements from the fundamental S-matrix. Remarkably, these are exactly the same equations that arose in the nested Bethe Ansatz. In other words, the coefficients  $B, C, D, E, F$  correspond to elements from the fundamental S-matrix and we again find (3.17) as the unique solution for  $M, N, h$ .

It is worthwhile to note that in this way, we have derived the complete fundamental S-matrix. This derivation differs fundamentally from the standard derivations [11, 8] since it depends crucially on the full Yangian symmetry. There might be a relation with [44] where the fundamental quantum S-matrix was also derived from Yangian symmetry.

To conclude, let us give the explicit solutions for  $L_0, L_1$ ,

$$\begin{aligned}
 L_0 &= \frac{g(y_1 - y_2)x_1^- x_2^-}{2i(y_1 - x_1^-)(y_2 - x_1^-)(y_1 - x_2^-)(y_2 - x_2^-)} \times \\
 &\times \left[ (y_1 + y_2) - \frac{x_1^- x_2^- x_1^+ x_2^+ (x_1^- + x_2^- + x_1^+ + x_2^+)}{2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+} - \frac{y_1 y_2 x_1^+ x_2^+}{2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+} \right. \\
 &\quad \left. \left\{ (x_1^+ + x_2^+ - x_1^- - x_2^-)(x_1^- x_2^- - x_1^+ x_2^+) - \left( \frac{1}{x_2^-} + \frac{1}{x_1^+} + \frac{1}{x_2^+} + \frac{1}{x_1^-} \right) (x_1^- x_2^- + x_1^+ x_2^+) \right\} \right] \\
 L_1 &= \frac{y_1 y_2 x_1^- x_2^-}{(y_1 - x_1^-)(y_2 - x_1^-)(y_1 - x_2^-)(y_2 - x_2^-)} \left[ (y_1 - y_2) + \frac{4ig^{-1}x_1^- x_2^- x_1^+ x_2^+}{2x_1^+ x_2^+ x_1^- x_2^- - x_1^- x_2^- - x_1^+ x_2^+} \right]
 \end{aligned} \tag{4.22}$$

Note that they are indeed manifestly symmetric under  $p_1 \leftrightarrow p_2$ .

**Bound states.** One might hope that it is possible to repeat the construction of previous section and find explicit elements of the bound state S-matrices again. However, when considering bound states, one encounters a difficulty. There is a new term, which is of the form  $W_{\alpha\beta 1^{\ell_1-2}}^{(i)}$ . This term behaves exactly like  $W_{21^{\ell_1-1}}^{(i)}$  and therefore it is hard to rewrite everything as in (4.19) in a unique way and read of S-matrix elements.

Nevertheless, we redo the procedure and try to match (4.16) to the obvious generalization of the two excitation ansatz

$$\begin{aligned}
 |\alpha\beta\rangle &= \sum_{k < l} \Psi_k(y_1)\Psi_l(y_2)W_{1^{\ell_1}}^{(1)} \dots W_{\alpha 1^{\ell_k-1}}^{(k)} \dots W_{\beta 1^{\ell_l-1}}^{(l)} \dots W_{1^{\ell_K}}^{(K)} + \\
 &+ \mathbb{S}^{\text{II}} \sum_{k < l} \Psi_k(y_1)\Psi_l(y_2)W_{1^{\ell_1}}^{(1)} \dots W_{\alpha 1^{\ell_k-1}}^{(k)} \dots W_{\beta 1^{\ell_l-1}}^{(l)} \dots W_{1^{\ell_K}}^{(K)} + \\
 &+ \epsilon^{\alpha\beta} \sum_k \Psi_k(y_1)\Psi_k(y_2)h(y_1, y_2, p_k)W_{1^{\ell_1}}^{(1)} \dots W_{21^{\ell_k-1}}^{(k)} \dots W_{1^{\ell_K}}^{(K)} + \\
 &+ \sum_k \Psi_k(y_1)\Psi_k(y_2)g(y_1, y_2, p_k)W_{1^{\ell_1}}^{(1)} \dots W_{\alpha\beta 1^{\ell_k-2}}^{(k)} \dots W_{1^{\ell_K}}^{(K)}. \tag{4.23}
 \end{aligned}$$

Or, restricted to two sites

$$\begin{aligned}
|\alpha\beta\rangle = & \{f(y_1, p_1)f(y_2, p_2)S(y_2, p_1) + Mf(y_2, p_1)f(y_1, p_2)S(y_1, p_1)\} W_{\alpha 1^{\ell_1-1}}^{(1)} W_{\beta 1^{\ell_2-1}}^{(2)} \\
& + Nf(y_2, p_1)f(y_1, p_2)S(y_1, p_1)W_{\beta 1^{\ell_1-1}}^{(1)} W_{\alpha 1^{\ell_2-1}}^{(2)} \\
& + g(y_1, y_2, p_1)f(y_1, p_1)f(y_2, p_1)W_{\alpha\beta 1^{\ell_1-2}}^{(1)} W_{1^{\ell_2}}^{(2)} \\
& + g(y_1, y_2, p_2)f(y_2, p_2)f(y_1, p_2)S(y_1, p_1)S(y_2, p_1)W_{1^{\ell_1}}^{(1)} W_{\alpha\beta\ell_2-2}^{(2)} \\
& + \epsilon^{\alpha\beta} h(y_1, y_2, p_1)f(y_2, p_1)f(y_1, p_1)W_{21^{\ell_1-1}}^{(1)} W_{1^{\ell_2}}^{(2)} \\
& + \epsilon^{\alpha\beta} h(y_1, y_2, p_2)f(y_2, p_2)f(y_1, p_2)S(y_2, p_1)S(y_1, p_1)W_{1^{\ell_1}}^{(1)} W_{21^{\ell_2-1}}^{(2)},
\end{aligned} \tag{4.24}$$

This is indeed possible and imposing symmetry of  $L_0, L_1$  as before provides equations that are uniquely solved by  $g(y_1, y_2, p_k) = \frac{\ell_k-1}{2\ell_k}(1 + M - N)$  and (3.17). These factor are, more or less, expected from fusion. Plugging these solutions back in  $L_0, L_1$ , we find the same functions  $L_0, L_1$  as in (4.22) but one has to bear in mind that now  $x^\pm$  parameterize bound state solutions.

For completeness, we give the explicit two particle wave function (restricted to two sites):

$$\begin{aligned}
|\alpha\beta\rangle = & \{f(y_1, p_1)f(y_2, p_2)S(y_2, p_1) + Mf(y_2, p_1)f(y_1, p_2)S(y_1, p_1)\} W_{\alpha 1^{\ell_1-1}}^{(1)} W_{\beta 1^{\ell_2-1}}^{(2)} \\
& + Nf(y_2, p_1)f(y_1, p_2)S(y_1, p_1)W_{\beta 1^{\ell_1-1}}^{(1)} W_{\alpha 1^{\ell_2-1}}^{(2)} \\
& + \frac{\ell_1-1}{2\ell_1}(1 + M - N)f(y_1, p_1)f(y_2, p_1)W_{\alpha\beta 1^{\ell_1-2}}^{(1)} W_{1^{\ell_2}}^{(2)} \\
& + \frac{\ell_2-1}{2\ell_2}(1 + M - N)f(y_1, p_2)f(y_2, p_2)S(y_1, p_1)S(y_2, p_1)W_{1^{\ell_1}}^{(1)} W_{\alpha\beta\ell_2-2}^{(2)} \\
& + \epsilon^{\alpha\beta} h(y_1, y_2, p_1)f(y_2, p_1)f(y_1, p_1)W_{21^{\ell_1-1}}^{(1)} W_{1^{\ell_2}}^{(2)} \\
& + \epsilon^{\alpha\beta} h(y_1, y_2, p_2)f(y_2, p_2)f(y_1, p_2)S(y_2, p_1)S(y_1, p_1)W_{1^{\ell_1}}^{(1)} W_{21^{\ell_2-1}}^{(2)},
\end{aligned} \tag{4.25}$$

with  $f, S, h, M, N$  as in (3.11), (3.17). By construction, this wave function satisfies (3.5) for any bound state S-matrix. Hence this solves our two excitation case. In particular one finds that also the level II S-matrix,  $\mathbb{S}^{\text{II}}$  is unchanged for bound states.

### 4.3 Bethe equations

By making use of coproducts and Yangian symmetry, we have found a way, independent of the explicit form of the S-matrix, to write down Bethe wave functions. This allowed us to find  $\mathbb{S}^{\text{II,I}}$  and we found that the level two S-matrix,  $\mathbb{S}^{\text{II}}$  remains unchanged. Hence all the higher level factors also remain unchanged. In other words, this yields that the Bethe

equations for any combination of bound states are given by:

$$\begin{aligned}
 e^{ip_k L} &= \prod_{l=1, l \neq k}^{K^I} \left[ S_0(p_k, p_l) \frac{x_k^+ - x_l^-}{x_k^- - x_l^+} \sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}} \right]^2 \prod_{\alpha=1}^2 \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^-}{x_k^+}} \\
 1 &= \prod_{l=1}^{K^I} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_k^-}{x_k^+}} \prod_{l=1}^{K_{(\alpha)}^{III}} \frac{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} + \frac{i}{g}}{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} - \frac{i}{g}} \\
 1 &= \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} + \frac{i}{g}}{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} - \frac{i}{g}} \prod_{l \neq k}^{K_{(\alpha)}^{III}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}},
 \end{aligned} \tag{4.26}$$

with

$$x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{2i\ell_k}{g}, \quad \frac{x_k^+}{x_k^-} = e^{ip_k}. \tag{4.27}$$

However, note that apart from the parameters  $x^\pm$ , the phase factor  $S_0(p_k, p_l)$  also implicitly depends on the considered bound states via [27]:

$$\begin{aligned}
 S_0(p_1, p_2) &= \left( \frac{x_1^-}{x_1^+} \right)^{\frac{\ell_2}{2}} \left( \frac{x_2^+}{x_2^-} \right)^{\frac{\ell_1}{2}} \sigma(x_1, x_2) \times \\
 &\quad \times \sqrt{G(\ell_2 - \ell_1)G(\ell_2 + \ell_1)} \prod_{l=1}^{\ell_1 - 1} G(\ell_2 - \ell_1 + 2l).
 \end{aligned} \tag{4.28}$$

The derived Bethe equations coincide with the equations one expects from a fusion procedure. This justifies a fusion procedure at the level of the Bethe ansatz equations.

Finally, we will present the Bethe equations in the mirror theory since they will have applications there for the TBA program. In the mirror theory the parameters  $a, b, c, d$  describing the symmetry algebra have the same dependence on  $x^\pm$  as in the original theory. However  $x^\pm$  are now dependent on the mirror momentum

$$x_i^\pm(\tilde{p}) = \frac{\ell_i}{2g} \left( -\sqrt{1 + \frac{4g^2}{\ell_i^2 + \tilde{p}^2}} \mp 1 \right) \left( -\frac{\tilde{p}}{\ell_i} - i \right). \tag{4.29}$$

To analyze bound states in the mirror theory it is more convenient to pick up a vacuum build up out of fermions ( $\mathfrak{sl}(2)$  sector). The derivation can be repeated for this case and the Bethe equations are given by [23]

$$\begin{aligned}
 e^{i\tilde{p}_k L} &= \prod_{l=1, l \neq k}^{K^I} S_0(\tilde{p}_k, \tilde{p}_l)^2 \prod_{\alpha=1}^2 \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{x_k^+ - y_l^{(\alpha)}}{x_k^- - y_l^{(\alpha)}} \sqrt{\frac{x_k^-}{x_k^+}} \\
 -1 &= \prod_{l=1}^{K^I} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_k^-}{x_k^+}} \prod_{l=1}^{K_{(\alpha)}^{III}} \frac{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} + \frac{i}{g}}{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} - \frac{i}{g}} \\
 1 &= \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} + \frac{i}{g}}{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} - \frac{i}{g}} \prod_{l \neq k}^{K_{(\alpha)}^{III}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}},
 \end{aligned} \tag{4.30}$$

where  $\tilde{p}$  is the mirror momentum and  $x^\pm$  are now given by (4.29). The  $-1$  on the left hand side is due to anti-periodic boundary conditions for the fermions [23]. For a detailed account of the mirror theory and its analytic properties we refer to [23].

## Acknowledgments

I am indebted to G. Arutyunov and S. Frolov for many valuable discussions. I would also like to thank A. Torrielli for comments on the manuscript. This work was supported in part by the EU-RTN network *Constituents, Fundamental Forces and Symmetries of the Universe* (MRTN-CT-2004-005104), by the INTAS contract 03-51-6346 and by the NWO grant 047017015.

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